Deviations, absolute values, mini-max:

Assume having the following constraints, where x1 and x2 are variables:

2x1 + 3x2 = 7

4x1 – x2 = 7

X1 + x2 = 5

X1 and x2 are free variables

If only the first 3 equations are considered: 2 equations and 2 unknowns (x1, x2) = (2,1)

With all 3 constraints, the problem is infeasible – can’t solve for x1, x2, introduce error term for each constraint, x3, x4, x5. (there are more equation than variable)

**Use all 3 constraints and the best approximate solution**

Minimizing the sum of square error (SSE)

Introducing x3, x4, and x5 to be the error terms for constraints, 1, 2, and 3

Objective:

Minimize (x3)^2 + (x4)^2 + (x5)^2

2x1 + 3x2 + x3 = 7

4x1 – x2 + x4 = 7

X1 + x2 + x5 = 5

The above formulation is non-linear as the objective function is a quadratic nonlinear program

Therefore, minimize the sum of the absolute value of error

Objective:

Minimize |x3| + |x4| + |x5| (which is still non-linear)

2x1 + 3x2 + x3 = 7

4x1 – x2 + x4 = 7

X1 + x2 + x5 = 5

All variables are “free” variables

This non-linear program can be transformed into a linear program

1. X3 = x3+ - x3-, x3+ and x3- are both non-negative
2. |x3| = x3+ + x3-

One of the pair = x3, the other is 0

For (1) and (2) both to be true, at least x3+ or x3- must be 0 (they can both be 0)

Minimize x3+ + x3- + x4+ + x4- + x5+ + x5-

2x1 + 3x2 + x3+ + x3- = 7

4x1 – x2 + x4+ + x4- = 7

X1 + x2 + x5+ + x5- = 5

X1 and x2 are free variables (neg, pos, or 0), x3+, x3-, x4+, x4-, x5+, x5- ≥ 0

The above formulation is linear, how to ensure that one of each pair of deviation variables will always be zero in optimal solution?

Weak argument: we are “manipulating” with the objective function to ensure that only one variable of the pair will be greater than 0 at any time in an optimal solution

Strong argument:

If x3+ = (1,0,0) then x3- = (-1,0,0), these 2 variables are linearly dependent, (perfectly colinear), hence only one of the variables from any pair can be in the solution

Objective:

Minimize maximum{(x3)^2 , (x4)^2 , (x5)^2}

2x1 + 3x2 + x3 = 7

4x1 – x2 + x4 = 7

X1 + x2 + x5 = 5

Minimize t

2x1 + 3x2 + x3 = 7

4x1 – x2 + x4 = 7

X1 + x2 + x5 = 5

All x variables are free variables

t >= 0

At optimal solution t\* will essentially hedge, and two out of three of (x3)^2, (x4)^2, (x5)^2 will be “tied” for the max squared error.

Minimize the maximum absolute deviation

Objective:

Minimize max {|x3|, |x4|, |x5|}

2x1 + 3x2 + x3 = 7

4x1 – x2 + x4 = 7

X1 + x2 + x5 = 5

All variables are “free” variables

Minimize t

Constraints:

2x1 + 3x2 + x3 = 7

4x1 – x2 + x4 = 7

X1 + x2 + x5 = 5

-t <= x3 <= t

-t <= x4 <= t

-t <= x5 <= t

all x variables are free variables

t >= 0

linear program and we avoided crated the extra x3+ and x3- variables